PHYSICS 232 | Final Examination | Name: Jake Bobowski
2012/13 Term 2 | | Student Number: Sol'ns

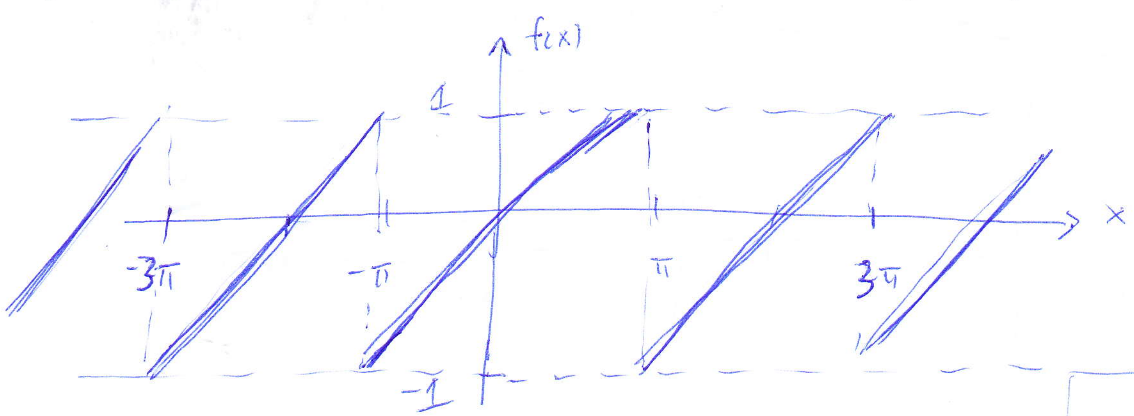
Free Response: Write out complete answers to the following questions. Show your work.

(10pts) **1.** Assume that the function $f(x)$ is periodic with period 2π such that $f(x+2\pi) = f(x)$. On the interval $-\pi < x < \pi$, $f(x)$ is given by $f(x) = x/\pi$.

(a) Sketch several periods of $f(x)$. Be sure to include scales for both the x- and y-axes of your plot. (2 marks)

(b) Find the Fourier series for this "sawtooth" function. Simplify your answers as much as possible. Write out the first five non-zero terms of the Fourier series. (8 marks)

(a)



(b)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{\pi}\cos nx\,dx$$

integrate by parts: $u = x$, $du = dx$, $dv = \cos nx\,dx$, $V = \frac{\sin nx}{n}$

$$a_n = \frac{1}{\pi^2}\left[\frac{x}{n}\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{n\pi^2}\left[\pi \cancel{\sin n\pi}^{0} + \pi\cancel{\sin(-n\pi)}^{0} + \frac{\cos nx}{n}\Big|_{-\pi}^{\pi}\right]$$

$$= \frac{1}{n^2\pi^2}\left[\cos n\pi - \cos(-n\pi)\right] = \frac{1}{n^2\pi^2}\left[\cos n\pi - \cos n\pi\right]$$

$$\therefore \boxed{a_n = 0 \;\forall n}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi^2}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi^2}x^2\Big|_{-\pi}^{\pi} = \frac{1}{2\pi^2}\left(\pi^2 - (-\pi)^2\right) = 0$$

10 pts

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{x}{\pi}\sin nx\,dx$$

$$u = x \qquad dv = \sin nx\,dx$$
$$du = dx \qquad v = -\frac{\cos nx}{n}$$

$$b_n = \frac{1}{\pi^2}\left[-\frac{x\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx\,dx \right]$$

$$= \frac{1}{n\pi^2}\left[-\pi\cos n\pi + (-\pi)\underbrace{\cos(-n\pi)}_{\cos n\pi} + \cancel{\frac{\sin nx}{n}}\Big|_{-\pi}^{\pi} \right] \xrightarrow{} 0$$

$$= \frac{1}{n\pi^2}\left[-2\pi\cos n\pi \right] = -\frac{2}{n\pi}\cos n\pi$$

n	$\cos n\pi$
1	-1
2	1
3	-1
4	1

$$\cos n\pi = (-1)^n$$

$$-\cos n\pi = (-1)(-1)^n = (-1)^{n+1}$$

$$\therefore b_n = \frac{2(-1)^{n+1}}{n\pi}$$

n	b_n
1	$2/\pi$
2	$-1/\pi$
3	$2/3\pi$
4	$-1/2\pi$
5	$2/5\pi$

$$\boxed{\therefore f(x) = \frac{2}{\pi}\sin x - \frac{1}{\pi}\sin 2x + \frac{2}{3\pi}\sin 3x - \frac{1}{2\pi}\sin 4x + \frac{2}{5\pi}\sin 5x - \dots}$$

(10pts) **2.** For each of the problems below, assume that you have measured $x \pm \sigma_x$ and $y \pm \sigma_y$. Take A and B to be known constants with negligible uncertainties and n to be an exact integer.

(a) If $f = A\,x + B\,y$, find an expression for σ_f. (2 marks)

(b) If $f = A\,x\,y^n$, find an expression for σ_f/f. Under what circumstances would the contribution of σ_x to σ_f be negligible? (3 marks)

(c) If $f = A^{B\,x}$, find an expression for σ_f/f. (2.5 marks)

(d) If $f = \ln\left[\sin^n(A\,x)\right]$, find an expression for σ_f. (2.5 marks)

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2$$

(a) $\sigma_f^2 = (A\sigma_x)^2 + (B\sigma_y)^2$

(b) $\sigma_f^2 = (Ay^n\sigma_x)^2 + (nAxy^{n-1}\sigma_y)^2$

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\cancel{Ay^n}\sigma_x}{\cancel{A}x\cancel{y^n}}\right)^2 + \left(\frac{n\cancel{Ax}\cancel{y^{n-1}}}{\cancel{Ax}\,y^{\cancel{n}}}\sigma_y\right)^2 = \boxed{\left(\frac{\sigma_x}{x}\right)^2 + \left(n\frac{\sigma_y}{y}\right)^2}$$

$\frac{\sigma_x}{x}$ term can be neglected if $\frac{\sigma_x}{x} \ll n\frac{\sigma_y}{y}$

(c) $f = A^{Bx}$ $\therefore \ln f = Bx\ln A$ $\therefore f = e^{(B\ln A)x}$

$$\frac{\partial f}{\partial x} = (B\ln A)e^{(B\ln A)x} = (B\ln A)e^{\ln A^{Bx}} = (B\ln A)A^{Bx} = (B\ln A)f$$

$\therefore \sigma_f^2 = ((B\ln A)f\sigma_x)^2$ $\therefore \left(\frac{\sigma_f}{f}\right)^2 = (B\ln A\,\sigma_x)^2$

$$\boxed{\frac{\sigma_f}{f} = (B\ln A)\sigma_x}$$

10 pts

(d) $f = \ln\left[\sin^n(Ax)\right] = n \ln\left[\sin(Ax)\right]$

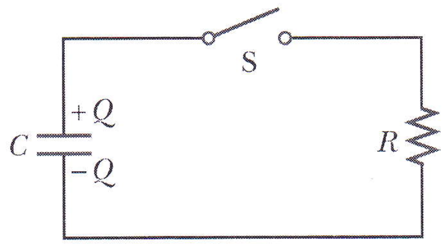
$$\frac{\partial f}{\partial x} = \frac{n}{\sin Ax} \frac{\partial}{\partial x}\left(\sin Ax\right) = \frac{n}{\sin Ax} A \cos Ax$$

$$= nA \cot Ax$$

$$\therefore \sigma_f^2 = \left(nA \cot Ax\right)^2 \sigma_x^2$$

$$\boxed{\sigma_f = \left|nA\cot(Ax)\right| \sigma_x}$$

(20pts) **3.** Consider the RC circuit shown in the figure. The capacitor is initially charged to voltage V_0. At $t = 0$ the switch is closed and the voltage across the capacitor is recorded as a function of time as shown in the table.



time (s)	V_C (V)
100	3.4 ± 0.2
200	2.3 ± 0.2
300	1.8 ± 0.2
400	1.1 ± 0.2

(a) The voltage across the capacitor is expected to evolve with time according to $V_C = V_0 e^{-t/\tau}$. Linearize this expression such that, using the data given above, the parameters V_0 and τ could be extracted from a linear fit. Clearly explain what you would plot and how the parameters would be extracted from the linear fit. (5 marks)

(b) Using the data given above, create a new table of X and $Y \pm \sigma_Y$. Where a plot of Y vs X is expected to produced a set of linear data as discussed in part (a). For this problem, assume that the uncertainty in time is negligible. (5 marks)

(c) Using the graph paper provided, plot your Y vs X data. Include the σ_Y as error bars. Clearly label your axes and provide a scale for both the $x-$ and $y-$axes. Don't make your plot tiny, use a large portion of the graph paper! Draw a straight line through your data. No calculations are necessary here, just use your best judgement. From your line, estimate V_0 and τ. No error estimates are required. (5 marks)

(d) Using your plot (data and line), estimate the value of χ^2. Clearly explain how your are determining χ^2. (5 marks)

(a) $V_c = V_0 e^{-t/\tau}$ $\qquad \ln V_c = \ln V_0 - \frac{t}{\tau}$

Plot $\underbrace{\ln V_c}_{y}$ vs $\underbrace{t}_{x}$ $\qquad$ slope $m = -\frac{1}{\tau}$

y-intercept $b = \ln V_0$.

20 pts

(b) $\cancel{Y} \to \ln V_c$ $\sigma_y = \dfrac{\sigma_{V_c}}{V_c}$

$X \to t$

$X_{(s)}$	Y
100	1.224 ± 0.059
200	0.833 ± 0.087
300	0.588 ± 0.11
400	0.095 ± 0.18

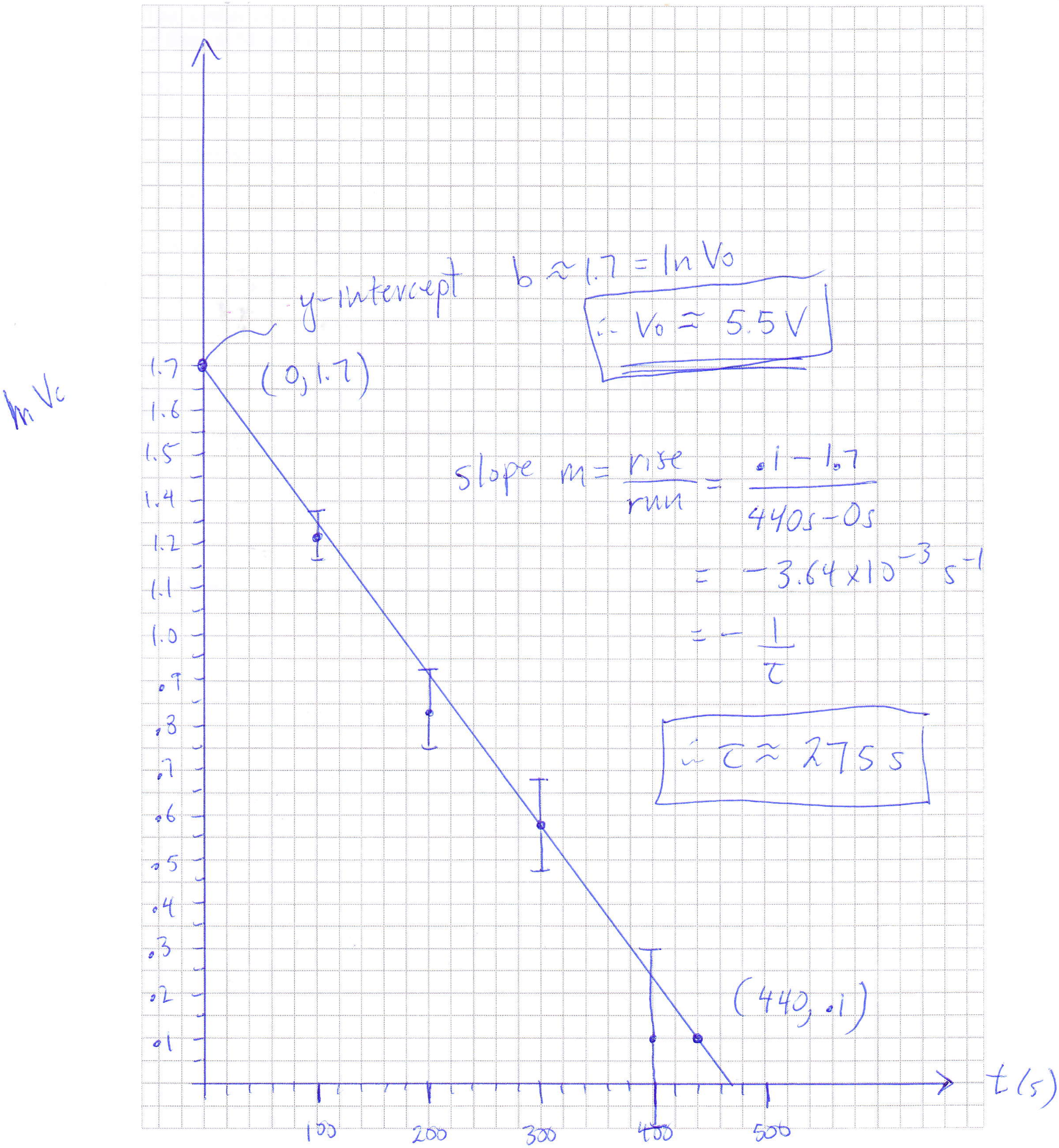
(d) $$\chi^2 = \sum_i \left(\frac{\cancel{y}_i - y(x_i)}{\sigma_i} \right)^2$$

i	$y_i - y(x_i)$	σ_i
1	$-.025$	$.059$
2	$-.075$	$.087$
3	0	$.11$
4	$.15$	$.18$

deviation of data pt from line.

$\chi^2 \simeq 1.62$

0 pts



y-intercept $b \approx 1.7 = \ln V_0$

$\boxed{\therefore V_0 \approx 5.5\,V}$

slope $m = \frac{\text{rise}}{\text{run}} = \frac{.1 - 1.7}{440s - 0s}$

$= -3.64 \times 10^{-3}\ s^{-1}$

$= -\frac{1}{\tau}$

$\boxed{\therefore \tau \approx 275\ s}$

0 pts

(10pts) **4.** The number of flaws in a fibre optic cable follows a Poisson distribution. The average number of flaws in 50 m of cable is 1.2. Recall that:

$$P_P = \frac{\mu^x}{x!}e^{-\mu}$$

(a) What is the standard deviation in the number of flaws in 50 m of cable? (1 mark)

(b) What is the probability of exactly three flaws in 150 m of cable? (3 marks)

(c) What is the probability of at least two flaws in 100 m of cable? (3 marks)

(d) What is the probability of exactly one flaw in the first 50 m of cable and four or fewer flaws in the next 200 m of cable? (3 marks)

(a) For Poisson $\sigma = \sqrt{\mu} = \sqrt{1.2} = \boxed{1.095}$

(b) $\mu_{50} = 1.2 \quad \therefore \mu_{150} = 3(1.2) = 3.6$

$P_3 = \frac{3.6^3}{3!}e^{-3.6} = \boxed{.212}$

(c) $\mu_{100} = 2.4$

$P_{\geq 2} = 1 - P_0 - P_1 \qquad P_0 = \frac{\mu^0}{0!}e^{-\mu} = e^{-\mu} = .091$

$P_1 = \frac{\mu^1}{1!}e^{-\mu} = .218$

$\boxed{\therefore P_{\geq 2} = 0.691}$

(d) $(P_{1 \text{ in } 50})(P_{\text{less than 4 in 200}})$ $\qquad P_{1 \text{ in } 50} = \frac{(1.2)^1}{x!}e^{-1.2} = .361$

$\mu_{200} = 4(1.2) = 4.8 \qquad P_{\text{less than 4}} = P_0 + P_1 + P_2 + P_3 + P_4$

$= 0.0032 + 0.0395 + .095 + .152 + .182$

$= .477$

10 pts

$$\therefore P_{net} = (.361)(.477) = \boxed{.172}$$

(10pts) **5.** Bits are sent over a communications channel in packets of 12. The probability of a bit being corrupted over this channel is 0.1 and such errors are independent.

(a) What is the probability that no more than 2 bits in a packet are corrupted? (4 marks)

(b) If 6 packets are sent over the the channel, what is the average number of packets that contain 3 or more corrupted bits? What is the spread or stand deviation in the number of packets containing 3 or more corrupted bits? (3 marks)

(c) If 6 packets are sent over the channel, what is the probability that at least one packet will contain 3 or more corrupted bits? (3 marks)

Binomial $P_B = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$

(a) $n=12 \quad p=0.1 \quad 1-p=0.9$

Prob. of zero or one or two bits corrupted.

$P_0 = (1-p)^{12} = .282$

$P_1 = 12p(1-p)^{11} = .377$

$P_2 = \frac{12.11}{2} p^2 (1-p)^{10} = .230$

$\boxed{.889}$

(b) For each packet, prob. of 3 or more corrupted bits is $1-.889$

$p' = .111$

For $n=6$ packets $\boxed{\mu = np' = .666}$ avg no. of packets w/ 3 or more corrupted bits if sample set of 6

$\sigma = \sqrt{npq} = \sqrt{np'(1-p')}$

$\boxed{\sigma = .769}$

10 pts

Ref

$\cancel{P_1 = \frac{\mu^1}{x!} e^{-\mu} =}$

$P_0 = \frac{\mu^0}{0!} e^{-\mu} = e^{-\mu} = e^{-.666} = .514$

prob. that none have 3 or more corrupted bits.

$\therefore \boxed{P_{\geq 3} = 1 - P_0 = .486}$

(c) Binomial

$P_0 = \frac{6!}{0!\,6!} p^0 (1-p')^6 = (1-p')^6 = .494$

$\therefore P_{\geq 3} = 1 - P_0 = \boxed{.506}$ at least one w/ 3 or more corrupted bits.

(10pts) **6.** (a) If repeated measurements of a quantity x are made and the uncertainty in each individual measurement σ is the same (like the oscillation period in Ruchhardt's experiment), then the mean is rather simply estimated via:

$$\mu = \frac{1}{N}\sum_i x_i.$$

Derive an expression for the uncertainty in the mean σ_μ. *Hint:* Use error propagation. (5 marks)

(b) Discuss what happens when the uncertainties in the individual measurements are not the same. How are the mean and the uncertainty in the mean modified? (5 marks)

(a) $\mu = \mu(x_1, x_2, \ldots, x_N)$

$$\sigma_\mu^2 = \left(\frac{\partial \mu}{\partial x_1}\sigma_1\right)^2 + \left(\frac{\partial \mu}{\partial x_2}\sigma_2\right)^2 + \cdots + \left(\frac{\partial \mu}{\partial x_N}\sigma_N\right)^2$$

$$\frac{\partial}{\partial x_j}\mu = \frac{\partial}{\partial x_j}\frac{1}{N}(x_1 + x_2 + \cdots + x_j + \cdots + x_N) = \frac{1}{N}$$

$$\therefore \sigma_\mu^2 = \left(\frac{\sigma_1}{N}\right)^2 + \left(\frac{\sigma_2}{N}\right)^2 + \cdots + \left(\frac{\sigma_N}{N}\right)^2$$

now all σ_i equal

$$\sigma_\mu^2 = \left(\frac{\sigma}{N}\right)^2 + \left(\frac{\sigma}{N}\right)^2 + \cdots + \left(\frac{\sigma}{N}\right)^2 = N\left(\frac{\sigma}{N}\right)^2 = \frac{\sigma^2}{N}$$

$$\therefore \boxed{\sigma_\mu = \frac{\sigma}{\sqrt{N}}}$$

10 pts

(b) When individual ~~meas.~~ σ_i not the same, then realize that prob. of meas. x_i in random meas. given by Gaussian dist'n

$$P_i \propto e^{-\left(\frac{x_i-\mu}{\sigma_i}\right)^2}$$

$\therefore$ Prob. of meas. x_1, ~~and~~ & x_2, & x_3 ... becomes

$$P = P_1 P_2 P_3 \cdots \propto e^{-\sum\left(\frac{x_i-\mu}{\sigma_i}\right)^2}$$

Max. P by minimizing $\sum\left(\frac{x_i-\mu}{\sigma_i}\right)^2 = \chi^2$

$$\frac{\partial \chi^2}{\partial \mu} = 0 = -2\sum \frac{x_i-\mu}{\sigma_i^2} \quad \text{or} \quad \sum \frac{x_i}{\sigma_i^2} = \sum \frac{\mu}{\sigma_i^2}$$

$$\therefore \mu \sum \frac{1}{\sigma_i^2} = \sum \frac{x_i}{\sigma_i^2}$$

meas. w/ smaller error more import. when calc. mean!

$$\text{or} \quad \boxed{\mu = \frac{\sum \frac{x_i}{\sigma_i^2}}{\sum \frac{1}{\sigma_i^2}}}$$

again use prop. error formula to find

$$\sigma_\mu^2 = \frac{1}{\sum \frac{1}{\sigma_i^2}}$$

$$\frac{\partial \mu}{\partial x_j} = \frac{\frac{1}{\sigma_j^2}}{\sum \frac{1}{\sigma_i^2}} \qquad \therefore \sigma_\mu^2 = \sum_j \left(\frac{\frac{1}{\sigma_j}}{\sum \frac{1}{\sigma_i^2}}\right)^2 = \frac{1}{\left(\sum \frac{1}{\sigma_i^2}\right)^2} \sum \left(\frac{1}{\sigma_j}\right)^2 = \frac{1}{\sum \frac{1}{\sigma_i^2}} \checkmark$$

(10pts) **7.** This problem will explore some aspects of fitting functions to datasets. Discuss/comment on the following points:

1
- What is the origin of χ^2?

2
- Why is minimizing χ^2 a useful method for extracting best-fit parameters from datasets?

3
- When doing weighted fits, why is that we assign the weights as $1/\sigma_i^2$ where σ_i is the uncertainty in the i^{th} data point $(x_i, y_i \pm \sigma_i)$?

4
- Why is it that for models/fit functions that are linear in the unknown parameters we can algebraically determine the best-fit values for the parameters, but for functions nonlinear in the parameters we have to resort to inexact methods such as a grid search?

1. When making comparison between dataset & model, prob. ~~that~~ of meas y_i at a part. x_i is Gaussian distributed

$$P_i \propto e^{-\left(\frac{y_i - y(x_i)}{\sigma_i}\right)^2}$$

μ

Prob. of obtaining particular data set for many diff. x_i is

$$P = P_1 P_2 \cdots P_N \propto e^{-\sum\left(\frac{y_i - y(x_i)}{\sigma_i}\right)^2}$$

maximize P by minimizing $\sum\left(\frac{y_i - y(x_i)}{\sigma_i}\right)^2 = \chi^2$

2. Minimizing χ^2 equiv. to max. prob. that model parameters ~~fits data set~~ describe the dataset.

$y = a_1 f_1(x) + a_2 f_2(x)$ for example.

minimize χ^2 w.r.t. to a_1 & a_2 to maximize P from 1.

3. The $\frac{1}{\sigma_i^2}$ weighting comes about because of appearance of $1/\sigma_i^2$ in Gaussian distn which is valid when have random fluctuations.

10 pts

4. To minimize χ^2 w.r.t. a_1 & a_2 requires

$$\frac{\partial}{\partial a_1}\chi^2 \quad \& \quad \frac{\partial}{\partial a_2}\chi^2$$

if y of the form $y = a_1 f_1(x) + a_2 f_2(x)$,

then it is possible to obtain, for example,

set of 2 equations & 2 unknowns (a_1 & a_2)

that can be easily solved.

B/c $\frac{\partial y}{\partial a_1} = f_1(x) + a_2 f_2(x)$ has no more a_1 dependence.

~~when~~

If y is nonlinear fcn of a_1 & a_2, like

$$y = a_1 \sin a_2 x,$$

then end up with things like

$\frac{\partial y}{\partial a_2} = x a_1 \cos a_2 x$ for which cannot

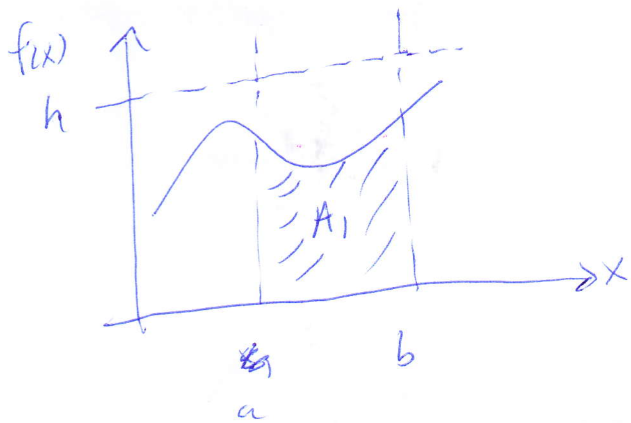
algebraicly solve for a_1 & a_2 from set

of 2 eq'ns.

(10[pts]) **8.** In class we discussed two Monte Carlo methods used to numerically evaluate definite integrals. Pick one of the two methods and outline how it works. Your answer should convey a conceptual understanding of the method and also outline how the method can be implemented. Use diagrams to aid your discussion.

Hit & Miss

want to find



$$\int_a^b f(x)\,dx = A_1$$

In hit & miss method generate random pairs of nos. (x_i, y_i) s.t.

$a \le x_i \le b$ & $0 \le y_i \le h$. If (x_i, y_i) beneath curve, call it a hit. Count no. of hits Z in total of N trials.

prob. that get hit will be $p = \frac{Z}{N}$ for N large.

Clearly p also given by ratio of A_1 to total area $h(b-a)$

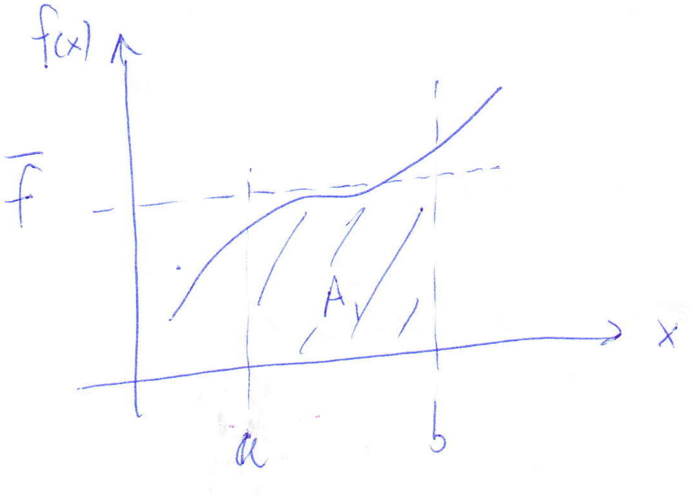
i.e. $p = \frac{A_1}{h(b-a)} = \frac{Z}{N}$ $\quad \boxed{\therefore A_1 = h(b-a)\frac{Z}{N}}$ (*) estimate of desired integral.

Implementation

1. general random coordinate
2. If $y_i < y(x_i)$, $Z = Z+1$
3. Repeat many times
4. Use (*) to find definite integral.

10 pts

$\bar{f}$ method.



want to ~~test~~ evaluate

$$A_1 = \int_a^b f(x)\,dx$$

Goal is to find equil. rectangular area

$$A_{eq} = \bar{f}(b-a) = A_1$$

Recall avg. value of fcn on $[a\ b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \therefore \int_a^b f(x)\,dx = (b-a)\bar{f}$$

In this method generate random x_i uniformly dist'd on $[a, b]$. For each x_i value, calc. $f_i = f(x_i)$

sum f_i values $\quad f_{tot} = \sum_{i=1}^{N} f_i$

Do N trials & find $\bar{f} = \frac{f_{tot}}{N} = \frac{1}{N}\sum_i f(x_i)$

$$\boxed{\therefore A_1 = \frac{(b-a)}{N}\sum_i f(x_i)} \quad \text{\#}$$

Implementation:

1. Generate random x_i
2. eval. $f(x_i)$ $\quad f_{tot} = f_{tot} + f(x_i)$
3. Repeat N times
4. est. integral using #